Advanced elastoplastic models

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Plan

Écrouissages non linéaire (1)

Écrouissages cinématique et isotrope nécessaires pour un comportement réaliste.

- cadre standard généralisé

$$f(\underset{\sim}{\sigma}, \underset{\sim}{X}, R) = J(\underset{\sim}{\sigma} - \underset{\sim}{X}) - \sigma_y - R + \frac{D}{2C} J^2(\underset{\sim}{X}) + \frac{R^2}{2Q}$$

$$\text{avec } J(\underset{\sim}{X}) = \left(\frac{3}{2} \underset{\sim}{X} : \underset{\sim}{X}\right)^{0,5}$$

- cadre associé

$$f(\underset{\sim}{\sigma}, \underset{\sim}{X}, R) = J(\underset{\sim}{\sigma} - \underset{\sim}{X}) - \sigma_y - R$$

Écrouissages non linéaire (2)

$$\dot{\underset{\sim}{\alpha}} = -\dot{\lambda}\frac{\partial f}{\partial \underset{\sim}{X}} = \left(\underset{\sim}{n} - \frac{3D}{2C}\underset{\sim}{X}\right)\dot{\lambda}$$

$$\dot{r} = -\dot{\lambda}\frac{\partial f}{\partial R} = \left(1 - \frac{R}{Q}\right)\dot{\lambda}$$

Variables d'état $\underset{\sim}{\alpha}$ et r, qui définissent les variables d'écrouissage $\underset{\sim}{X}$ et R :

$$\Psi = ... + \frac{1}{2}bQr^2 + \frac{1}{3}C\underset{\sim}{\alpha} : \underset{\sim}{\alpha}$$

$$\underset{\sim}{X} = \frac{2}{3}C\underset{\sim}{\alpha} \qquad ; \qquad R = b\,Qr$$

- multiplicateur plastique $\equiv$ vitesse de déformation plastique cumulée mais $r \neq p$
- *dans le cas où les coefficients sont constants* :

$$\dot{\underset{\sim}{X}} = \frac{2}{3}C\dot{\underset{\sim}{\varepsilon}}^p - D\underset{\sim}{X}\dot{p}$$

$$\dot{R} = b(Q - R)\dot{p} \qquad \text{soit } R = Q(1 - \exp(-b\,p))$$

Energie dissipée

$$\Phi_1 = \underset{\sim}{\sigma} : \underset{\sim}{\dot{\varepsilon}}^p - R\,\dot{r} - \underset{\sim}{X} : \underset{\sim}{\dot{\alpha}}$$
$$= \left(\underset{\sim}{\sigma} : \underset{\sim}{n} - R + \frac{R^2}{Q} - \underset{\sim}{X} : \underset{\sim}{n} + \frac{D}{C} J^2(\underset{\sim}{X}) \right) \dot{\lambda}$$

$$\underset{\sim}{\sigma} : \underset{\sim}{n} - \underset{\sim}{X} : \underset{\sim}{n} = J(\underset{\sim}{\sigma} - \underset{\sim}{X})$$

$$\Phi_1 = \left(J(\underset{\sim}{\sigma} - \underset{\sim}{X}) + \frac{R^2}{Q} + \frac{D}{C} J^2(\underset{\sim}{X}) \right) \dot{\lambda}$$
$$= \left(f + \sigma_y + \frac{R^2}{2Q} + \frac{D}{2C} J^2(\underset{\sim}{X}) \right) \dot{\lambda}$$

Dissipation :
- $f\,\dot{\lambda}$, dissipation visqueuse ;
- $\sigma_y\,\dot{\lambda}$, dissipation (“de friction”) liée au seuil initial ;
- termes quadratiques, non–linéarité de l’écrouissage ;

Énergie bloquée

(≡ variation d'énergie libre) :

$$\dot{\Psi} = R\,\dot{r} + \underset{\sim}{X} : \dot{\underset{\sim}{\alpha}}$$
$$= Q(1 - e^{-bp})\,e^{-bp}\,\dot{p} + \underset{\sim}{X} : (\underset{\sim}{n} - \frac{3D}{2C}\underset{\sim}{X})\,\dot{p}$$

(*récupérable ou non ?*)

Elementary hardening variables

$$f(\underset{\sim}{\sigma}, \underset{\sim}{X}, R) = J(\underset{\sim}{\sigma} - \underset{\sim}{X}) - R - \sigma_y \qquad f(\sigma, X, R) = |\sigma - X| - R - \sigma_y$$

- Isotropic hardening depend on p, the *accumulated plastic strain* defined as :

$$\dot{p} = \left(\frac{2}{3}\underset{\sim}{\dot{\varepsilon}}^p : \underset{\sim}{\dot{\varepsilon}}^p\right)^{1/2} = |\dot{\varepsilon}^p|$$

- Linear kinematic hardening depend on ε^p, the *present plastic strain*

- Nonlinear kinematic hardening depend on α, defined as :

$$\underset{\sim}{\dot{\alpha}} = (\underset{\sim}{n} - D\underset{\sim}{\alpha})\dot{p} \qquad \dot{\alpha} = (sign(\dot{\varepsilon}^p) - D\alpha)\,\dot{p}$$

asymptotic value of α *= 1 / D*

Hardening variables

- Isotropic hardening :

$$R = Q(1 - \exp(-bp))$$

- Linear kinematic hardening :

$$X = C\varepsilon^p$$

- Nonlinear kinematic hardening ($X = C\alpha$) :

$$\dot{X} = (C - DX sign(\dot{\varepsilon}^p))\,\dot{\varepsilon}^p$$

for tensile loading :

$$X = (C/D)(1 - \exp(-D\varepsilon^p))$$

Integration of the nonlinear kinematic model

- Tension, first branch

$$X = \frac{C}{D}\left(1 - exp(-D\varepsilon^p)\right)$$

- Tension going branch, from (ε_0^p, X_0) to (ε_1^p, X_1)

$$X_1 = \frac{C}{D} + (X_0 - \frac{C}{D})\, exp(-D(\varepsilon_1^p - \varepsilon_0^p))$$

  - Since $X_1 = -X_0$, the variation is

$$\frac{\Delta X}{2} = \frac{C}{D}\tanh(D\Delta^p/2)$$

- 1D ratchetting

$$\delta\varepsilon^p = \frac{C}{D}\left(\frac{(C/D)^2 - (\sigma_{min} + \sigma_y)^2}{(C/D)^2 - (\sigma_{max} - \sigma_y)^2}\right)$$

Flow

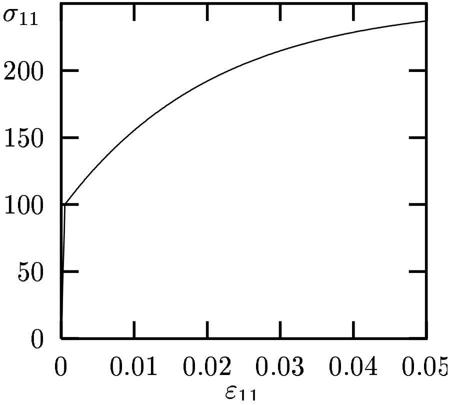
- Viscoplastic flow :

$$\dot{\varepsilon}^p = \left\langle \frac{|\sigma - X| - R - \sigma_y}{K} \right\rangle^n sign(\sigma - X)$$

- Tensile loading :

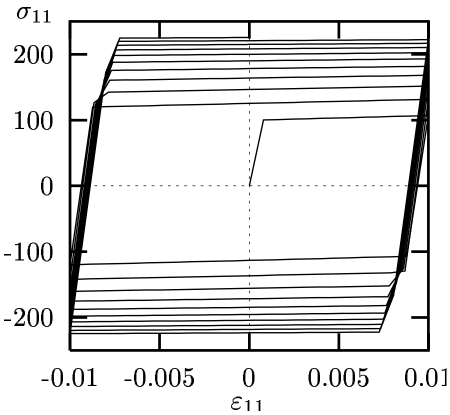
$$\sigma = \sigma_y + Q(1 - \exp(-b\varepsilon^p)) + \frac{C}{D}(1 - \exp(-D\varepsilon^p)) + K(\dot{\varepsilon}^p)^{1/n}$$

Tensile test



	Isotrope	Cin NL
σ_y	100	100
Q	150	0
b	50	0
C	0	7500
D	0	50

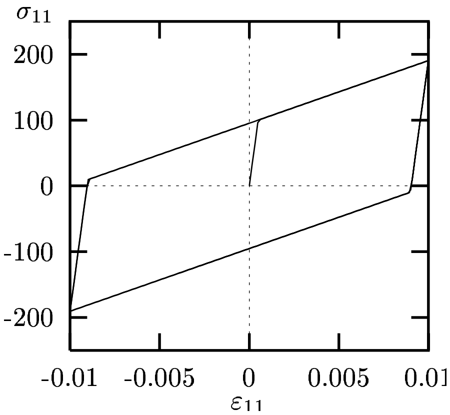
Cyclic Iso



σ_y	100
Q	150
b	5
C	0
D	0

a. Isotropic hardening

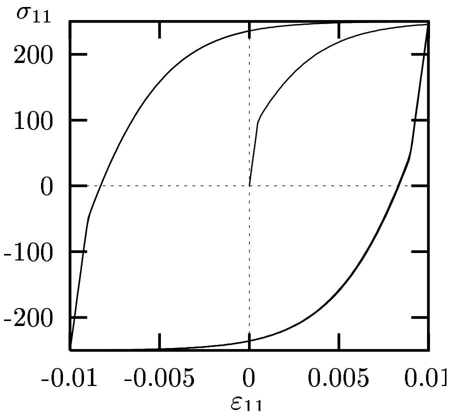
Cyclic Lin Kin



σ_y	100
Q	0
b	0
C	10000
D	0

b. Linear kinematic hardening

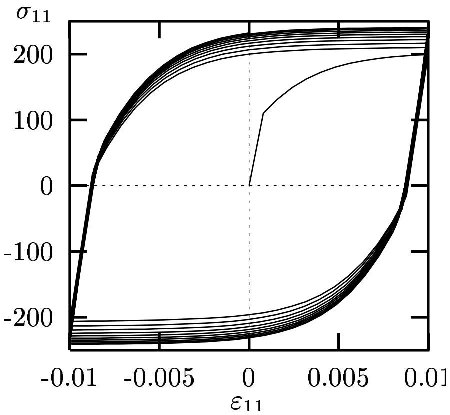
Cyclic Nonlin Kin



σ_y	100
Q	0
b	0
C	60000
D	400

c. Nonlinear kinematic hardening

Cyclic Iso + Nonlin Kin



σ_y	100
Q	50
b	5
C	40000
D	400

d. Isotropic + Nonlinear kinematic hardening

A few classical models in viscoplasticity

$$\dot{\varepsilon}^{vp} = \left\langle \frac{|\sigma| - \sigma_y}{K} \right\rangle^n sign(\sigma) \quad , \quad \dot{\varepsilon}^{vp} = \dot{\varepsilon}_0 \left\langle \frac{|\sigma|}{\sigma_y} - 1 \right\rangle^n sign(\sigma)$$

$$\dot{\varepsilon}^{vp} = \left(\frac{\sigma}{\sigma_{eq}} - 1 \right)^n$$

$$\dot{\varepsilon}^{vp} = A\mathrm{sh}\left(\frac{|\sigma|}{K} \right) sign(\sigma)$$

$$\dot{\varepsilon}^{vp} = \left\langle \frac{|\sigma - X| - R - \sigma_y}{K} \right\rangle^n sign(\sigma - X)$$

– kinematic hardening (X is the *internal stress*) ;
– isotropic hardening ($R + \sigma_y$ is the *friction stress*) ;
– hardening on the viscous stress (K is the *drag stress*).

Rôle of each coefficient

R0	σ_y, initial yield stress
Q	cyclic hardening or softening
b	convergence rate to Q
C/ D	asymptotic value of X
D	convergence rate to C/D
K	viscous stress for $\dot{\varepsilon}^p = 1 s^{-1}$
n	$\rightarrow 1$ for high temperature

- for $\sigma_y = R = X = 0$, Norton model
- for $\sigma_y = R = 0$, no threshold (non linear viscoelasticity)
- for small K, no more viscous effect ($\rightarrow$ time independent plasticity)

Phenomenological aspects

- Modeling of R_m (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$R_m = R0 + Q + (C/D) + K \times 0.001^{1/n}$$

- Modeling of $R_{0.2}$ (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$R_{0.2} = R0 + Q(1 - \exp(-0.002 \times b)) + (C/D)(1 - \exp(-0.002 \times D)) + K \times 0.001^{1/n}$$

- Modeling of the cyclic hardening curve (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$\Delta\sigma/2 = R0 + Q + (C/D)\tanh(D\Delta\varepsilon^p/2) + K \times 0.001^{1/n}$$

- Secondary creep rate

$$\dot{\varepsilon}^p = \left\langle \frac{\sigma - (C/D) - R - R0}{K} \right\rangle^n$$

- Asymptotic stress in relaxation

$$\sigma_\infty = R0 + Q + (C/D)$$

Chaboche's model

```
***behavior gen_evp
**elasticity isotropic young 160000.  poisson 0.3
**potential gen_evp ep
*criterion mises
*flow norton              K    300.     n    7.
*kinematic linear         C  10000.
*kinematic nonlinear      C 180000.     D  600.
*isotropic nonlinear      R0   300.     Q  100.  b   10.
***return
```

$$\begin{aligned}
\sigma &= R0 + Q(1 - e^{-b\varepsilon^p}) && \textit{isotropic} \\
&+ H\varepsilon^p && \textit{kinematic} \\
&+ C/D\,(1 - e^{-D\varepsilon^p}) && \textit{kinematic} \\
&+ K\,(\dot{\varepsilon}^p)^{1/n} && \textit{viscous}
\end{aligned}$$

$\rightarrow$ 8 material parameters

A list of typical effects related to hardening

- Type of hardening, cyclic hardening/softening
- Shape of the loop
- Redistribution of the mean stress under non symmetric prescribed strain
- Elastic shakedown, plastic shakedown, ratchetting
- Unsymmetric behaviour
- Anisotropy
- Creep-Plasticity interaction
- Recovery
- Aging
-

A list of possible flow rules

- Power function

$$\dot{p} = \left\langle \frac{f}{K} \right\rangle^n$$

- Double power function

$$\dot{p} = \left\langle \frac{f}{K_1} \right\rangle_1^n + \left\langle \frac{f}{K_2} \right\rangle_2^n$$

- Hyperbolic

$$\dot{p} = \dot{\varepsilon}_0 \left(\sinh \left\langle \frac{f}{K} \right\rangle^n \right)^m$$

- Sellars & Tegart

$$\dot{p} = \dot{\varepsilon}_0 \left(\sinh \left\langle \frac{f}{K} \right\rangle \right)^m$$

A list of possible isotropic hardening rules

- By point
- Linear

$$R = R_0 + Hp$$

- Non linear

$$R = R_0 + Q(1 - \exp(-bp))$$

- Power law

$$R = R_0 + K(e_0 + p)^n$$

- Linear/non linear

$$R = R_0 + Hp + Q(1 - \exp(-bp))$$

- Non linear/sum

$$R = R_0 + \sum_i^N Q_i(1 - \exp(-b_i p))$$

A list of possible kinematic hardening rules

- Non linear “with phi”

$$\dot{\underset{\sim}{\alpha}} = (\underset{\sim}{n} - \phi(p)\frac{3D}{2C}\underset{\sim}{X})\dot{p}$$

with $\phi(p) = \phi_m + (1 - \phi_m)\exp(-\omega p)$

- Non linear with “radial fading memory”

$$\dot{\underset{\sim}{\alpha}} = \underset{\sim}{n} - (\eta \underset{\sim}{I} + \frac{2}{3}(1 - \eta)\underline{n} \otimes \underline{n})\frac{3D}{2C}\underset{\sim}{X}$$

- Non linear with a threshold

$$\dot{\underset{\sim}{\alpha}} = (\underset{\sim}{n} - \phi(\underset{\sim}{X})\frac{3D}{2C}\underset{\sim}{X})\dot{p}$$

with $\phi(\underset{\sim}{X}) = \left\langle \frac{DJ(\underset{\sim}{X}) - \omega}{1 - \omega} \right\rangle^{m_1} \frac{1}{DJ(\underset{\sim}{X})^{m_2}}$

Static recovery

- On isotropic hardening

$$\dot{r} = \left(1 - \frac{R}{Q}\right)\dot{p} - \frac{1}{bQ}\left(\frac{R}{M}\right)^m$$

with $R = bQr$

- On kinematic hardening

$$\dot{\underset{\sim}{\alpha}} = \dot{\underset{\sim}{\varepsilon}}^p - D\underset{\sim}{X}\dot{p} - \frac{J(\underset{\sim}{X})}{M}^m \frac{\underset{\sim}{X}}{J(\underset{\sim}{X})}$$

  - 1D version, steady state :

$$\dot{X}_s = 0 = C\dot{\varepsilon}^p_s - DX_s\dot{p}_s - \left(\frac{|X_s|}{M}\right)^m \mathrm{sign}(X_s)$$

  - From ε^p_s in the previous equation, find the stress vs X

$$\sigma = X_s + \sigma_y + K\dot{\varepsilon}_s^{p^{1/n}}$$